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Plasma Simulation with Few Particles*

GEORG KNORR

Department of Physics and Astronomy,
The University of Iowa, Iowa City, Iowa 52240

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A numerical method of plasma simulation is described which allows one to simulate Vlasov plasmas with very few degrees of freedom. Linear effects can be simulated by an amount of information corresponding to less than 100 particles, nonlinear effects require less than 500 "particles" in one spatial and one velocity dimension. The method is based on replacing the infinite real eigenvalue spectrum of the free streaming terms of the Vlasov equation by a finite discrete spectrum. An imaginary damping term is artificially added to the eigenvalues, and, thus, recurrence effects are minimized.

1. INTRODUCTION

This paper deals with methods and techniques for an economical simulation of plasmas, which can be described by the Vlasov equation

$$\frac{\partial f(x, v, t)}{\partial t} + v \frac{\partial f}{\partial x} + \frac{e}{m} E(x, t) \frac{\partial f}{\partial v} = 0. \tag{1}$$

The characteristics of Eq. (1) are given by

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = \frac{e}{m} E(x, t). \tag{2}$$

It is a well known theorem that integration of the set of ordinary differential Eq. (2) is equivalent to the solution of the partial differential Eq. (1). Most of the plasma simulation has been done by integrating many thousands of the characteristics of Eq. (2) which are the particle trajectories. However, it should be remembered that the mathematical equivalence of Eq. (1) and (2) is numerically true only if a continuum of particle trajectories is simulated. If due to limitation of computer storage and time only a finite number of particles can be simulated, the

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solution of Eq. (2) agrees with the solution of Eq. (1) to the zeroth order in the parameter $(n\lambda_d^3)^{-1}$, where λ_d is the Debye length and n is the number of simulated particles per unit volume.

Good results are obtained when about 10^4 particles are used in one dimension (one coordinate in x and one in v). If we want to represent the distribution function in phase space in two dimensions (two spatial and two velocity coordinates) with similar accuracy we have to use $(10^4)^2$ particles because only then does the phase space cube which is represented by one particle have the same length. This (ideal) requirement cannot be satisfied by existing computers, and, consequently, simulations in more dimensions cannot claim as high an accuracy as in the one dimensional case. Introduction of a magnetic field complicates the situation somewhat, and it will not be discussed here. In any case it is meaningful to study methods which might reduce the number of "particles" or rather degrees of freedom with which we describe a plasma and yet preserve the physical relevant effects which we want to simulate.

The particular difficulty with the Vlasov equation is that it tends to develop filaments in time; superimposed over a relatively smooth part of the distribution function is another part with an ever increasing slope in the v-direction. This has been described by Armstrong, Harding, Knorr, and Montgomery [1]. In difference schemes of the Vlasov equation the filamentation causes an early deterioration of the simulation [2]. If a Hermite polynomial representation of velocity space is used, the filamentation causes recurrence phenomena due to the cutoff of an infinite matrix [3]. If a particle simulation is used the filamentation manifests itself in large fluctuations and stochastic noise [4].

If it were possible to suppress the filamentations which have been shown to contribute little to the macroscopic quantities like density, electric field, and energy, it might be possible to do plasma simulation with less computational effort. This would not only help to make computations cheaper but would also allow us to simulate more complex situations and to do plasma simulation in three dimensions.

A particle simulation of plasmas is essentially a statistical description of the distribution function; its values are defined by the number of particles in a representative unit of phase space volume. Such a description is not very efficient because one has to store N particles in order to describe one value of the distribution function with error of $N^{-1/2}$.

If transform methods are used to solve Eq. (1), a representation in spatial Fourier modes has been used earlier [5]. However, the product $E(x, t) \cdot \partial f(x, v, t)/\partial v$ in Eq. (1) is then converted into a convolution sum, which increases the computational effort considerably. This has been realized by Nuehrenberg [6] who designed a code which utilized configuration space but transformed velocity space. The difficulty of an appropriate cutoff in velocity space which gets rid of filamentation, but keeps the important parts of the distribution function, remained unsolved.

For the case of Hermite functions the problem of cutoff has been investigated by Joyce, Knorr, and Meier [3]. It has been found that a continuous eigenvalue spectrum is transformed into a discrete finite spectrum by the cutoff procedure. A prescription for the cutoff procedure has been given which attributes an imaginary damping decrement to the eigenvalues and, thus, practically eliminates recurrence effects. No effort, however, was made to use this result to construct an efficient plasma simulation code. Considerations for such a code are presented in the following. The objections raised by Nuehrenberg against the use of parabolic cylinder functions still remain valid; we have to expect a slow convergence to the correct solution with the number of Hermite polynomials used. But no other set of classical orthogonal functions possesses such a simple expression for the derivative of a polynomial. At least at this stage of the development of the code, this argument outweighs the disadvantages of the Hermite polynomials.

The outline for the rest of this paper is as follows: in Section 2 we derive a Hermite representation which minimizes the number of operations; Section 3 is devoted to stability questions; Section 4 discusses initialization procedures; and Section 5 presents questions concerning the nonlinear force term. Section 6 presents results of the method. Finally, a summary and the conclusions are found in Section 7.

2. Representation in Terms of Hermite Polynomials

Similar to Ref. [1], we represent the v-dependence of the distribution function by a series of orthogonal polynomials

$$f(x, v, t) = \frac{1}{(2\pi)^{1/2}} \sum_{\nu=0}^{\infty} h_\nu b_\nu(x, t)\, He_\nu(v) \exp(-\tfrac{1}{2}v^2). \tag{3}$$

As mentioned, the orthogonal set of polynomials used is essentially arbitrary. Recently Lewis [7] used nonclassical polynomials for a somewhat similar problem. Constant factors h have been added in Eq. (3). They are arbitrary and will be adjusted for a maximum of numerical convenience. Inserting ansatz [3] into Eq. (1) results in

$$\frac{\partial}{\partial t} b_\nu + \frac{\partial}{\partial x} b_{\nu-1} \frac{h_{\nu-1}}{k_\nu} + \frac{\partial}{\partial x} b_{\nu+1} \frac{h_{\nu+1}(\nu+1)}{h_\nu} - E(x, t)\, b_{\nu-1} \frac{h_{\nu-1}}{h_\nu} = 0. \tag{4}$$

We adjust the h_ν in such a way as to make the two coefficients $h_{\nu-1}/h_\nu$ and $(\nu+1)/h_\nu/h_{\nu+1}$ equal. Then the equation will contain only one coefficient

$$\rho_\nu = h_{\nu-1}/h_\nu\,,$$

which is efficient for numerical speed. For the thus defined ρ_ν the recursion relation

$$\rho_\nu \rho_{\nu+1} = \nu + 1$$

holds, and Eq. (4) becomes

$$\dot{b}_\nu(x, t) + \rho_\nu \left[\frac{\partial}{\partial x}(b_{\nu-1} + b_{\nu+1}) - E(x, t)\, b_{\nu-1}\right] = 0. \tag{5}$$

The choice of ρ_0 is arbitrary. A convenient choice is $\rho_0 = 1$. Similarly we choose $h_0 = 1$ which entails $h_1 = 1$, $h_2 = \frac{1}{2}$, etc.

Sometimes it is convenient and enhances numerical accuracy if the distribution function is split into a homogeneous part and a perturbation, similar to the analytical approach to the linearized Vlasov equation. We put

$$f(x, v, t) = f_0(v) + f_1(x, v, t)$$

with

$$f_1(x, v, t) = \frac{1}{(2\pi)^{1/2}} \sum_{\nu=0}^{\infty} b_\nu h_\nu He_\nu(v) \exp(-\tfrac{1}{2}v^2) \tag{6}$$

and

$$\frac{\partial f_0(v)}{\partial v} = \frac{1}{(2\pi)^{1/2}} \sum_{\nu=0}^{\infty} c_\nu h_{\nu-1} He_\nu(v) \exp(-\tfrac{1}{2}v^2). \tag{7}$$

We then obtain, instead of Eq. (5),

$$\dot{b}_\nu + \rho\nu \left[\frac{\partial}{\partial x}(b_{\nu-1} + b_{\nu+1}) + E(b_{\nu-1} - c_\nu)\right] = 0. \tag{8}$$

For example, we choose

$$f_0 = (2\pi)^{-1/2} \exp(-\tfrac{1}{2}v^2),$$

and find $c_\nu = -\delta_{1\nu}$, δ being the Kronecker symbol.

For the distribution which is subject to a two-stream instability,

$$f_0(v) = (2\pi)^{-1/2} \frac{1 + \beta v^2}{1 + \beta} \exp(-\tfrac{1}{2}v^2),$$

we obtain $c_1 = -1$, $c_3 = 2\beta/(1 + \beta)$, and all other coefficients equal to zero.

The systems (5) or (8) are infinite and cannot be handled by a computer. If they are truncated by setting $b_N(x, t) = 0$, the nature of the system is radically changed, as pointed out in Ref. [3]. If we set $E \equiv 0$ for the moment, decaying solutions of Eq. (1) become almost periodic solutions of Eq. (5) because the continuous eigenvalue spectrum of Eq. (1) is replaced by a discrete, finite number of eigenvalues of the truncated system.

This can be seen easily. With

$$b_\nu(x, t) = (-i)^\nu a_\nu \exp(ikx + \Lambda t), \tag{9}$$

system (5) is transformed into the algebraic system

$$\Lambda a_\nu + k\rho_\nu(-a_{\nu-1} + a_{\nu+1}) = 0.$$

We put $a_\nu = \zeta_\nu H_\nu$, where $\zeta_\nu/\zeta_{\nu+1} = i\rho_\nu$, and obtain

$$H_{\nu+1} = -\frac{i\Lambda}{k} H_\nu - \nu H_{\nu-1}. \tag{10}$$

This is the recursion equation for Hermite polynomials. Thus,

$$H_\nu = He_\nu\left(-\frac{i\Lambda}{k}\right),$$

$He_\nu(x)$ being a Hermite polynomial [8].

The system is finite if $He_N(-i\Lambda/k) = 0$ or $\Lambda_\mu = ika_\mu{}^N$, where the $a_\mu{}^N$ are the zeros of the Nth Hermite polynomial with $\mu = 1, 2, 3, \ldots, N$.

In order to make the solutions of the truncated system more similar to the true solution of Eq. (1), we require that all eigenvalues Λ_μ have a negative imaginary part. This is equivalent to the condition

$$He_N\left[\frac{-i(\Lambda + i\lambda)}{k}\right] = 0, \tag{11}$$

which can be rewritten as

$$He_N\left(\frac{-i\Lambda}{k}\right) = -\sum_{l=1}^{N}\left(\frac{\lambda}{k}\right)^l \binom{N}{l} He_{N-l}\left(\frac{-il}{k}\right). \tag{12}$$

In an infinite matrix a disturbance would simply travel to higher and higher indices. The "boundary condition" (12) avoids that a disturbance reaching the boundary of the truncated matrix suffers a reflection, and returns to the lower indices which represent macroscopic quantities, unabatedly. Rather the *disturbance interferes with an artificially created image which causes an exponential decay of the original disturbance.*

We make use of Eq. (12) to damp the filamentation of the Vlasov equation. With Eq. (10) and (9) we can rewrite Eq. (12) as

$$a_N = -\sum_{l=1}^{N}\left(-\frac{\lambda}{k}\right)^l \binom{N}{l} (\rho_{N-1}, \rho_{N-2}, \ldots, \rho_{N-l})^{-1} a_{N-l}, \tag{13}$$

and, with the aid of relation (9), we obtain for the b's

$$b_N = -\sum_{l=1}^{N} \frac{\sigma_{N-l}}{k^l} i^{-l} b_{N-l}, \tag{14}$$

where

$$\sigma_{N-l} = (-\lambda)^l \binom{N}{l} (\rho_{N-1}, \rho_{N-2}, \ldots, \rho_{N-l})^{-1}.$$

For sufficiently small λ the first few terms in the sum will already give a good approximation to b_N. Let us assume that the first M terms suffice. Then

$$(ik)^M b_N = -\sum_{l=1}^{M} (ik)^{M-l} \sigma_{N-l} b_{N-l}. \tag{15}$$

Going from the k into the x-representation, we find the following differential equation for b_N:

$$b_N^{(M)}(x, t) = -\sum_{l=1}^{M} \sigma_{N-l} b_{N-l}^{(M-l)} \equiv g(x). \tag{16}$$

The upper index in parenthesis characterizes the order of the derivative. This equation can easily be solved using a fast Fourier transform.

The question arises: What will happen if M, the number of terms in Eq. (16), is chosen too small. How will the damping of the different eigenvalues be affected? Computer experiments showed that the damping decrements of the different eigenvalues will deviate from their assigned value λ, some of them towards smaller damping. Thus, the recurrence effects will be increased. Under ordinary circumstances, however, 8–10 terms suffice to keep the deviation below a few percent.

The propagation of a disturbance along the columns of the matrix is similar to the propagation of an electromagnetic wave along a telegraph line. We can ask for the speed of propagation and, in particular, calculate the time after which a reflected disturbance appears again at the low-index side of the matrix.

A one-to-one correspondence of the Hermite polynomials with the terms in the power transform had been established in Ref. [3]. The speed of a disturbance along the y-component of the characteristic function is clearly

$$\frac{dy}{dt} = k. \tag{17}$$

One term of the power transform (compare Ref. [3, Eq. (13)])

$$y^p \exp(-\tfrac{1}{2}y^2)$$

has its maximum at $\nu^{1/2}$. Therefore, $a_\nu(t)$ takes on its maximum approximately when

$$\nu^{1/2} - kt = 0.$$

Considering ν for the moment as a continuous parameter we obtain for the velocity V_ν along the ν-index of the b-matrix

$$V_\nu = 2k\nu^{1/2}. \tag{18}$$

The velocity increases for larger ν and is proportional to the wave number. The recurrence time follows as

$$T_R = 2N^{1/2}/k, \tag{19}$$

when N is the number of Hermite polynomials.

An example for the damping of the recurrence effect is shown in Fig. 1. The

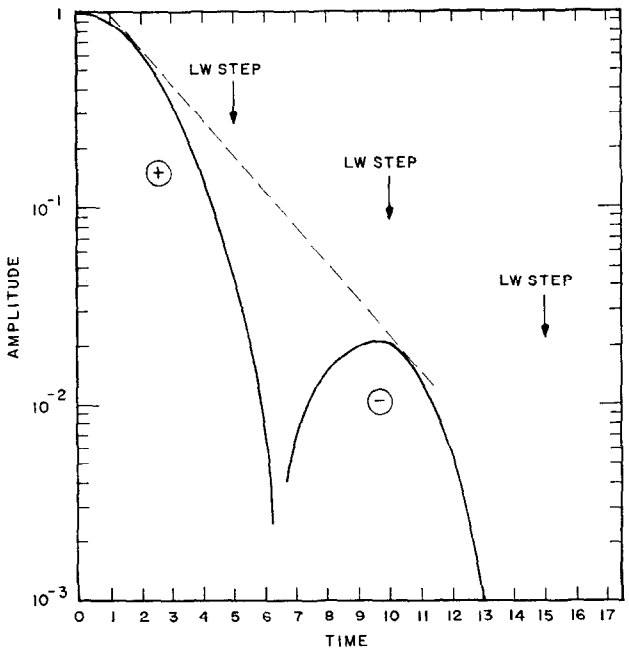

FIG. 1. The successful damping of the recurrence when integrating Eq. (1) with $E \equiv 0$ using a matrix of ten Hermite polynomials and eight points in x space. The initial decay is proportional to $\exp(-\frac{1}{2}k^2t^2)$. The recurrence at $t = 10$ is reduced to 2% of the original amplitude, due to a cutoff of the form Eq. (20). Putting $b_n(x, t) \equiv 0$ would have resulted in a recurrence comparable with the initial amplitude. The initial condition is $f(x, v, 0) = (2\pi)^{-1/2} \exp(-\frac{1}{2}v^2) \cos kx$, with $k = \frac{1}{2}$.

initial parabolic decay for $0 < t < 6$ on an exponential scale is what one expects for the free-streaming exact solution. After $t = 6$ the solution grows again and reaches about 2% of the initial value at $t = 10$.

3. Stability of Finite Difference Scheme

The system (5) is hyperbolic. A recent comparison of the available difference schemes for hyperbolic equations by Morton [9] has shown that the leapfrog scheme is superior to first order, Lax–Wendroff and Crank–Nicolson schemes. The modes have no damping and the phase representation is better than that of the other schemes. In addition, it is fast. A disadvantage is that the leapfrog method contains a parasitic mode of propagation which is also not damped.

We investigate now the stability of the leapfrog scheme when applied to Eq. (5) with $E \equiv 0$. With

$$\mathbf{b} = (b_0, b_1, \ldots) \qquad \text{and} \qquad \mathbf{R} = \begin{pmatrix} 0 & \rho_0 & 0 & 0 & \cdots \\ \rho_1 & 0 & \rho_1 & 0 & \cdots \\ \cdots & & & & \end{pmatrix}, \tag{20}$$

we write the free streaming part of the system (8) as

$$\frac{\partial}{\partial t}\mathbf{b} = -\mathbf{R}\frac{\partial \mathbf{b}}{\partial x}. \tag{20}$$

The centered difference scheme is

$$\mathbf{b}^{n+1}(j) - \mathbf{b}^{n-1}(j) = -\frac{\Delta t}{\Delta x}\mathbf{R}[\mathbf{b}^{n+1/2}(j + \tfrac{1}{2}) - \mathbf{b}^{n+1/2}(j - \tfrac{1}{2})]. \tag{21}$$

Considering normal modes, $b^n(j) = b^n \exp(ikj\,\Delta x)$, this becomes

$$\mathbf{b}^{n+1} = \mathbf{b}^{n-1} - 2i\frac{\Delta t}{\Delta x}\sin(\tfrac{1}{2}k\,\Delta x)\,\mathbf{R}\mathbf{b}^{n+1/2}. \tag{22}$$

As the leapfrog method is equivalent to a two-step method, we have to write Eq. (22) also for the time step $(n + 3/2)$.

$$\mathbf{b}^{n+3/2} = \mathbf{b}^{n+1/2} - ia\mathbf{R}\cdot\mathbf{b}^{n+1},$$

where $a = 2\,\Delta t/\Delta x \sin \tfrac{1}{2}k\,\Delta x$ or using Eq. (22)

$$\mathbf{b}^{n+3/2} = (\mathbf{I} - a^2\mathbf{R}^2)\,\mathbf{b}^{n+1/2} - ia\mathbf{R}\mathbf{b}^n. \tag{23}$$

In order to diagonalize $\mathbf{R}$ we multiply Eqs. (22) and (23) on the right with the polar matrix $\mathbf{P}$ of $\mathbf{R}$ and on the left with $\mathbf{P}^{-1}$ and make use of the relation

$$\mathbf{P}^{-1}\mathbf{R}\mathbf{P} = \Lambda, \qquad \Lambda_{ij} = \lambda_i\,\delta_{ij}.$$

The λ_i are the eigenvalues of $\mathbf{R}$. Equations (22) and (23) are transformed into

$$\begin{pmatrix} \mathbf{c}^{n+1} \\ \mathbf{c}^{n+3/2} \end{pmatrix} = \mathbf{G} \cdot \begin{pmatrix} \mathbf{c}^{n} \\ \mathbf{c}^{n+1/2} \end{pmatrix} \tag{24}$$

with

$$\mathbf{G} = \begin{pmatrix} \mathbf{I} & -ia\mathbf{\Lambda} \\ -ia\mathbf{\Lambda} & \mathbf{I} - a^2\mathbf{\Lambda}^2 \end{pmatrix}$$

and $\mathbf{c} = \mathbf{P}^{-1}\mathbf{b}\mathbf{P}$. $\mathbf{G}$ is the amplification matrix. To find the eigenvalues we multiply the first row of the determinant $\| \mathbf{G} - \mathbf{I}\lambda \|$ with $ia\lambda$ and add it to the $(n + 1)$ row, which has been multiplied with $(1 - \lambda)$. The determinant, thus, becomes triangular, and we obtain

$$\| \mathbf{G} - \mathbf{I}\lambda \| = \prod_{\nu=1}^{N} (1 - 2\lambda + a^2\lambda_\nu^2\lambda + \lambda^2) = 0. \tag{25}$$

The solution is

$$\lambda = 1 - \tfrac{1}{2}a^2\lambda_\nu^2 \pm ia\lambda_\nu(1 - \tfrac{1}{4}a^2\lambda_\nu^2)^{1/2}.$$

λ lies on the unit circle if $\frac{1}{4}a^2\lambda_\nu^2 < 1$. The condition for stability is, therefore,

$$\frac{\Delta t}{\Delta x} < |\lambda_N|^{-1}, \tag{26}$$

where λ_N is the largest eigenvalue of $\mathbf{R}$.

We can write the solution of Eq. (24) as $\lambda = \exp(\pm i\chi)$ with $\sin\chi = \frac{1}{2}a\lambda_\nu$ as one can easily corroborate by insertion. It follows that λ_ν must be real, otherwise one of the two solutions will necessarily lie outside the unit circle and the system will be unstable. This remark becomes important if we include in Eq. (20) the damping of the solutions due to the "boundary condition" (16), by adding time symmetric terms. This corresponds in the stability analysis to making the eigenvalues of $\mathbf{R}$ complex and our difference scheme will always be unstable.

However, if the last row of the matrix is not taken at time step $(n + \frac{1}{2})$, but at the time step n, the resulting system has stable solutions. In addition, this choice keeps the system explicit. The order of the damping term is now of first order in Δt only. But the damping term has been introduced artificially anyhow, so this cannot be considered of any importance.

The stability analysis with damping terms becomes quite involved. Only for a 2×2 matrix

$$\dot{y}_1 = \frac{\partial}{\partial x} y_2 ,$$

$$\dot{y}_2 = \frac{\partial}{\partial x} y_1 - dy_2 ,$$

the calculation could be done explicitly, and we find that the stability limit is given by

$$\Delta t\, d < 4\left[1 - \left(\frac{\Delta t}{\Delta x}\right)^2\right].$$

Thus, the damping decrement determines the stability limit in addition to $\Delta t/\Delta x$. Qualitatively similar observations have been made for the general case of N equations.

4. Initializing of the Leapfrog Scheme

The two-level leapfrog scheme has to be initialized properly. Ideally the initializing procedure should make the amplitude of the parasitic mode exactly zero. However, it appears that this is impossible without a fairly complex iteration procedure. Thus, a two-level Lax–Wendroff scheme was designed which accomplished an initialization of second order in Δt.

Even with a second-order initialization it was observed that the two levels of the leapfrog scheme tended to drift apart. An averaging of the two levels can be done in a simple way only to first order in Δt. As the Lax–Wendroff scheme was available from the initialization it was used to initialize the leapfrog scheme continuously after a number of time steps before the drifting of the two meshes becomes apparent. In actual runs the Lax–Wendroff initialization was repeated every 20, 40, or 80 time steps, depending on the magnitude of the time step. This procedure is not difficult to implement and gives satisfactory results.

5. The Force Term

When the electric field is included the second-order difference scheme takes the following form:

$$\begin{aligned} b_\nu^{n+1}(m) &= b_\nu{}^n(m) - \rho\nu\,\frac{\Delta t}{\Delta x}\{\delta[b_{\nu-1}^{n+1/2}(m) + b_{\nu+1}^{n+1/2}(m)] \\ &\quad + \Delta x\, E^{n+1/2}(m)\widetilde{[b_{\nu-1}^{n+1/2}(m)} + \delta_{1\nu}]\} = 0, \\ b_\nu^{n+3/2}(m+\tfrac{1}{2}) &= b_\nu^{n+1/2}(m+\tfrac{1}{2}) - \rho_\nu\,\frac{\Delta t}{\Delta x}\{\delta[b_{\nu-1}^{n+1}(m+1) + b_{\nu+1}^{n+1}(m+1)] \\ &\quad + \Delta x\, E^{n+1}(m+\tfrac{1}{2})\widetilde{[b_{\nu-1}^{n+1}(m+\tfrac{1}{2})} + \delta_{1\nu}]\} = 0. \end{aligned} \tag{27}$$

The operator δ is defined by

$$\delta b(m) = b(m + \tfrac{1}{2}) - b(m - \tfrac{1}{2}).$$

Poisson's equation becomes

$$E^{n+1/2}(m + 1) - E^{n+1/2}(m) = \Delta x \, b_1^{n+1/2}(m + \tfrac{1}{2}). \tag{28}$$

A tilde over a quantity indicates that the quantity is not directly available but must be approximated, e.g.

$$\widetilde{b(m)} = \tfrac{1}{2}[b(m + \tfrac{1}{2}) + b(m - \tfrac{1}{2})].$$

The stability properties of the system do not seem to be affected by the addition of the force term.

We discuss now the physical conservation theorems and how they are represented in the difference scheme. The conservation of mass follows from the first equation of system (27). We obtain

$$\sum_m b_0^{n+1}(m) = \sum_m b_0{}^n(m). \tag{29}$$

From the second equation we obtain

$$\sum_m b_1^{n+1}(m) = \sum_m b_1{}^n(m) + \Delta x \sum_m E^{n+1/2}(m) \, \tfrac{1}{2}[b_0^{n+1/2}(m + \tfrac{1}{2}) + b_0^{n+1/2}(m - \tfrac{1}{2})].$$

The momentum is constant if the last term vanishes. But using Eq. (28) we can write

$$\begin{aligned}\sum_m E^{n+1/2}(m)[b_0^{n+1/2}(m + \tfrac{1}{2}) + b^{n+1/2}(m - \tfrac{1}{2})] \\ = \frac{1}{\Delta x} \sum_m E^{n+1/2}(m)[E^{n+1/2}(m + 1) - E^{n+1/2}(m)] \\ + \frac{1}{\Delta x} \sum_m E^{n+1/2}(m)[E^{n+1/2}(m) - E^{n+1/2}(m - 1)] = 0.\end{aligned}$$

For the energy no exact conservation has been found from Eq. (27).

Nuehrenberg [6] used in his scheme the Maxwell equation and derived a simple prescription to obtain energy conservation. We tried the same here and obtained exact energy conservation. However, the momentum was no longer conserved which lead eventually to an instability. This approach was, therefore, abandoned.

6. Results

The ideas elaborated in the preceding sections have been incorporated into a code which is correct to fourth order in Δx and second order in Δt. The following examples have been studied: (a) almost linear Landau damping, (b) nonlinear Landau damping, and (c) the symmetric two-stream instability.

A typical case for the behavior of the electric field for almost linear Landau damping is shown in Fig. 2. The initial condition is

$$f(x, v, 0) = (2\pi)^{-1/2} \exp(-\tfrac{1}{2}v^2)(1 + A \cos kx) \tag{29}$$

with $A = 0$ and $k = 0.1$. A stable standing wave is produced. The initial amplitude is still large enough that the damping decrement decreases until about $t = 20$ in dimension less units of $(\omega_{pe})^{-1}$. This behavior is familiar from earlier work [1, 2]. The integration was run until $t = 80$ and from $t > 20$ on the Landau damping was constant. No sudden blow-up was observed which could be compared with Lewis' [7] method, because all the recurrence phenomena were sufficiently damped. Also plotted is the second harmonic. It exhibits beat-like oscillations on a time interval of 10. This is due to recurrences. From Eq. (18) follows that disturbances from higher modes travel faster along the matrix and they are less damped when

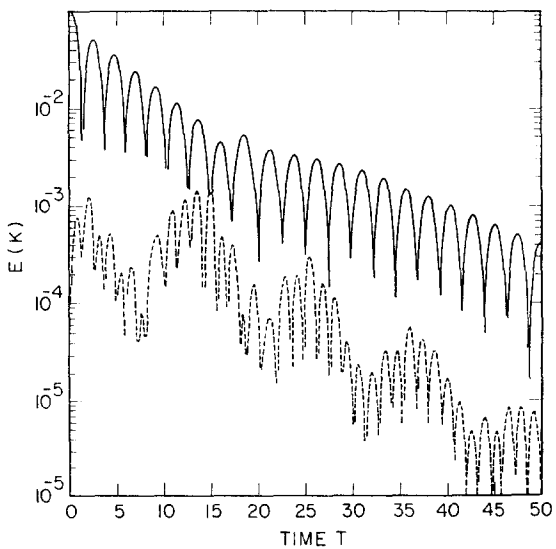

FIG. 2. Almost linear Landau damping for an initial condition $f(x, v, 0) = (2\pi)^{-1/2} \exp(-\frac{1}{2}v^2)(1 + A \cos kx)$ with $A = 0.1$ and $k = 0.5$. The amplitude is still large enough so that the damping decrement decreases in time, —— mode $k = 0.5$, – – – – mode $k = 1$.

they recur. Energetically this is irrelevant because the second mode is almost always one order of magnitude or more less than the first mode. This simulation is equivalent to the use of 480 "particles" as far as storage is concerned.

Reducing the amplitude A in Eq. (29) results in strictly linear Landay damping. It can successfully simulated with this code with as few as 80 particles.

The effect of increasing the initial amplitude A to $A = \frac{1}{2}$ is shown in Fig. 3.

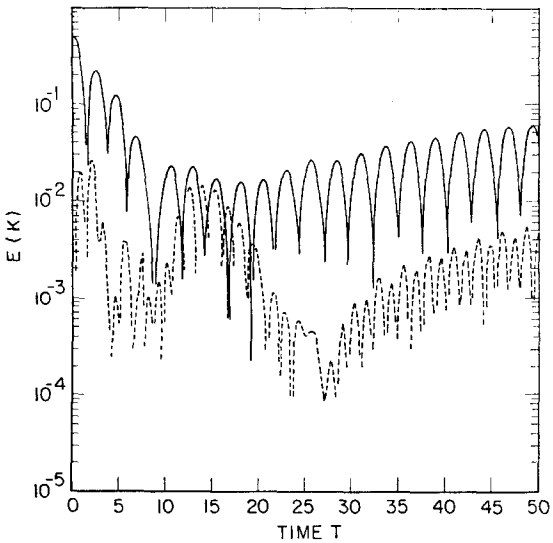

FIG. 3. Strongly nonlinear Landau damping for the same initial condition as in Fig. 2 but with $A = 0.5$ and $k = 0.5$, —— mode $k = 0.5$, - - - -

After a much stronger damping than according to the linear theory and a decay by roughly $1\frac{1}{2}$ orders of magnitude the amplitude grow again. This effect has also been observed earlier [6, 10]. Again we observe in the second mode a recurrence maximum at $t \approx 15$, which might have caused some distortions of the solution because it is comparable with the first mode. The first mode settles at a fairly constant level, contrary to the case of Nuehrenberg which shows a sudden decay after $t = 40$. In Nuehrenberg's case this is just the time after which uncontrollable and unphysical disturbances which enter through the open boundaries of his transformed velocity space can influence the electric field. We have collected computational parameters of the cases discussed here. Compared with the other cases the energy conservation is relatively bad. This is due to the fact that only eight points in x have been used. This simulation corresponds to the use of 240 particles.

Finally we consider the symmetric two-stream instability with the initial condition

$$f(x, v, 0) = (2\pi)^{-1/2} v^2 \exp(-\tfrac{1}{2}v^2)(1 + A \cos kx)$$

with $a = 0.05$ and $k = \frac{1}{2}$. A typical solution is shown in Figs. 4a and 4b. This instability has been studied by Denavit and Kruer [11], who compared particle codes with Vlasov codes. Figure 4a gives the magnitude of the first two modes as a function of time. Initially there is a mixture of several solutions of the linear dispersion equation. It follows that the exponential growth, saturation, and slower oscillations of the electric field are due to the trapping of particles. In Fig. 4b the

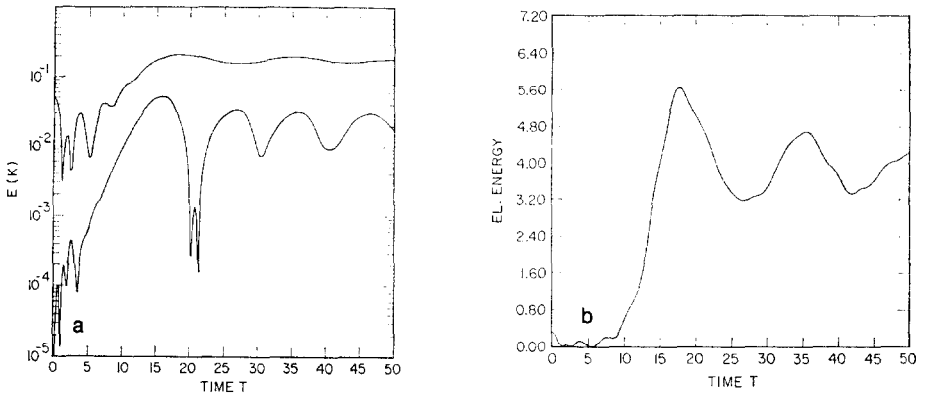

FIG. 4. A two-stream instability with the initial condition $f(x, v, 0) = (2\pi)^{-1/2} v^2 \exp(\frac{1}{2}v^2)$ $(1 + A \cos kx)$ with $A = 0.05$ and $k = \frac{1}{2}$. (a) Logarithmic plot of the electric fields for the modes $k = \frac{1}{2}$ and $k = 1$. The first mode is always larger than the second. (b) Plot of the total energy versus time. The units are the same as in Ref. [1]: length in λ_0, time in ω_{pe}^{-1}, velocity in v_{thermal}. In the figure, ten times the electric energy per unit length is plotted.

total electric energy is plotted linearly in time. This plot has all the physical features of Fig. 3 of Ref. [11]. It has been obtained with a remarkably small number of degrees of freedom, namely 480. The unoptimized code required 2 min 18 sec on an IBM 360/65 computer. We do not claim a comparable accuracy as Denavit and Kruer, but we want to demonstrate only that our method yields meaningful physical results with a minimum of information stored in the computer and very reasonable computation time. A substantial reduction of the number of degrees of freedom will reduce computation time for a 2D code because of the scaling laws, even if there may not yet be such a reduction in the 1D case.

7. Discussion

We have shown that familiar nonlinear effects can be reproduced from an integration of the Vlasov equation with a code which utilizes some new principles. The important feature is not that nonlinear effects can be recovered but that the computational effort is quite low.

Phase space is represented in our method by N points, which we call "particles" because they require the same storage as a particle code with N particles. We are able to run this code with less than 500 particles. The computation time per particle is of the order of 250 μsec for an unoptimized FORTRAN code on a IBM 360/65. This is comparable with the time for similar particle codes. Appreciable savings in computer time and cost are accomplished. This makes two-dimensional plasma simulation cheaper and more complicated situations can be simulated.

On the other hand, three-dimensional plasma simulation becomes more feasible with more economical codes. According to an estimate by Gazday, Canosa, and Armstrong [12] three dimensional plasma simulation will be possible in 1984 if an assumed extrapolated increase in computer speed and storage capacity materializes. The authors did not consider new principles and methods which might make feasible simulation in three dimensions at an earlier date.

What further improvements can be expected which would further increase the speed of the code? In paragraph 3 it was reported that the cutoff damping terms influence the stability behavior. Sometimes this is inconvenient because it makes the time step quite small for a given amount of damping. Preliminary experiments indicate that this problem can be solved by an impicit scheme.

Another inconvenient limitation is the dependence of the time step on the largest eigenvalue according to Eq. (26). If the number of Hermite polynomials is increased a smaller time step is required simultaneously. This restriction might be removed by using a different orthogonal system of polynomials over a finite interval.

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